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Exact minimizer for the couple (L^∞, BV) and the one-dimensional analogue of the Rudin–Osher–Fatemi model

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Abstract

We provide a simple algorithm that constructs an exact minimizer for the E-functional

$$E(t, f; L^\infty, BV) = \inf_{\|g\|_{L^\infty} \le t} \|f - g\|_{BV}.$$

Here L^∞, BV stand for the space of bounded functions and the space of functions with bounded variation on the interval $[a, b]$, respectively. As a corollary we obtain the following formula for the K-functional

$$K(N, f; BV, L^\infty) \backsim \sup_{a \le x_0 \le \cdots \le x_N \le b} \sum_{i=1}^{N} |f(x_i) - f(x_{i+1})|.$$

We also discussed the connection between the results and the Rudin–Osher–Fatemi denoising model.

Keywords: Exact minimizer; E-functional; K-functional; Rudin–Osher–Fatemi model

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1. Introduction and preliminaries

In image processing an important role is played by the Rudin–Osher–Fatemi (ROF) denoising model (see [6]). Suppose that we observe a noisy image

$$f = f_* + \eta,$$

where f_* is the initial image and η is the "noise". The ROF model suggests taking as an approximation to the initial image f_* the function $f_t \in BV$ which minimizes the functional

$$L_{2,1}(t, f; L^2, BV) = \inf_{g \in BV} (\|f - g\|_{L^2}^2 + t\,\|g\|_{BV}),$$

where BV is a space of functions of bounded variation on the rectangular domain $\Omega \subset \mathbb{R}^2$. When the following estimate of the noise is known

$$\|\eta\|_{L^2} \le \varepsilon,$$

the so-called Morozov discrepancy principle (see [9]) suggests choosing $t > 0$ such that $\|f - f_t\|_{L^2} = \varepsilon$. Let us explain the underlying idea of the Morozov principle from the point of view of interpolation theory. It follows from

$$\begin{aligned} \varepsilon^2 + t\,\|f_t\|_{BV} &= \|f - f_t\|_{L^2}^2 + t\,\|f_t\|_{BV} = L_{2,1}(t, f; L^2, BV) \\ &\le \|f - f_*\|_{L^2}^2 + t\,\|f_*\|_{BV} \le \varepsilon^2 + t\,\|f_*\|_{BV} \end{aligned}$$

that

$$\|f_t\|_{BV} \le \|f_*\|_{BV}\,.$$

Moreover,

$$\|f_t - f_*\|_{L^2} \le \|f_t - f\|_{L^2} + \|f - f_*\|_{L^2} \le 2\varepsilon.$$

Therefore for any $\theta \in (0, 1)$ we have

$$\|f_t - f_*\|_{L^2}^{1-\theta}\,\|f_t - f_*\|_{BV}^{\theta} \le 2\varepsilon^{1-\theta}\,\|f_*\|_{BV}^{\theta}\,.$$

So if the real interpolation space $(L^2, BV)_{\theta,1}$ (see [3,5,11]) is continuously embedded in a Banach space X, $(L^2, BV)_{\theta,1} \hookrightarrow X$, then we have immediately the inverse problem error estimate

$$\|f_t - f_*\|_X \le c\varepsilon^{1-\theta}\,\|f_*\|_{BV}^{\theta}\,.$$

Note that for the realization of this procedure two steps are needed: given $f \in L^2$ and $t > 0$, first we need to construct the minimizer f_t and then we repeat the first step until we find $t > 0$ such that $\|f - f_t\|_{L^2} = \varepsilon$.

The most difficult part is to find the minimizer f_t. Nowadays there exist several methods for constructing f_t approximately. It is possible to show that f_t is a solution of some nonlinear system of PDE and then to use an iterative procedure to solve it. This is probably the most popular method. However the problem is a typical approximation problem and it seems interesting to find f_t by purely approximation methods.

Such attempts have been also made. For example, in [7] it is shown that the Haar wavelets can be used for this purpose. Moreover, in the more recent paper [2], it is shown that the Haar wavelets can be used even for the n-dimensional analog of the Rudin–Osher–Fatemi model, i.e. for the couple $\left(L^{\frac{n}{n-1}}, BV\right)$. Unfortunately in both of these papers what is constructed is a *near-minimizer*, not an exact minimizer. For example, for $n = 2$ the method only gives

$$\|f - f_t\|_{L^2}^2 + t\,\|f_t\|_{BV} \le cL_{2,1}(t, f; L^2, BV)$$

with some numerical constant $c \ge 1$ and not $c = 1$. There is another approach developed in [1]. It is based on piecewise constant approximation and the Besicovitch covering theorem, but it also leads to a near-minimizer for the couple (L^2, BV).

In [2], it is shown by means of an example that the Haar wavelets are not appropriate for $n = 1$, i.e. for the couple (L^∞, BV). In this case the space BV is just a space of functions of bounded variations on the interval $[a, b]$ defined by the seminorm

$$\|f\|_{BV} = \sup_{N\in\mathbb{N}} \sup_{a\le x_0\le\cdots\le x_N\le b} \sum_{i=1}^{N} |f(x_i) - f(x_{i-1})|\,.$$

Our main goal in this note is to show that if $f \in C[a, b]$ then there exists a simple method for constructing an *exact* minimizer for the functional

$$E(t, f; L^\infty, BV) = \inf_{\|g\|_{L^\infty}\le t} \|f - g\|_{BV}\,.$$

Continuity of f allows us to use elementary and clear arguments.

Then we turn our attention to the K-functional

$$K(N, f; BV, L^\infty) = \inf_{g\in L^\infty} \{\|f - g\|_{BV} + N\|g\|_{L^\infty}\}.$$

Using the result for the E-functional, we derive the following interesting formula

$$K(N, f; BV, L^\infty) \backsim \sup_{a\le x_0\le\cdots\le x_N\le b} \sum_{i=1}^{N} |f(x_i) - f(x_{i+1})|\,.$$

In this equivalence, the constants are independent of $f \in C[a, b]$ and $N \in \mathbb{N}$. This formula can be also derived from results in the paper [4].

Formulae for the E- and the K-functional are established in Section 2. As it is well-known, real interpolation spaces can be defined by using K, E or L functionals. However there are only a few cases when it is possible to determine these functionals exactly or to construct near-minimizers with small constants. See, for example, the papers [10,8] and the books [3,5,11].

Our results do not have consequences in image denoising because they do not refer to the two-dimensional case. Nevertheless, they allow to give a one-dimensional model which is analogous to the ROF model. Moreover, since we work with the E-functional instead of the L-functional, we can realize the Morozov discrepancy principle in one step instead of two. This is shown in Section 3.

2. The algorithm

Let f be a continuous function on the closed interval $[a, b]$ and $t > 0$ be fixed. If

$$\frac{\max_{x\in[a,b]} f(x) - \min_{x\in[a,b]} f(x)}{2} \le t$$

then the choice

$$f_t(x) = \frac{\max_{x\in[a,b]} f(x) + \min_{x\in[a,b]} f(x)}{2}$$

yields $\|f - f_t\|_{L^\infty} \le t$ and $\|f_t\|_{BV} = 0$. So in this case

$$E(t, f; L^\infty, BV) = \inf_{\|g\|_{L^\infty}\le t} \|f - g\|_{BV} = 0.$$

Subsequently, we consider only the non-trivial case when

$$\frac{\max_{x\in[a,b]} f(x) - \min_{x\in[a,b]} f(x)}{2} > t.$$

Put $x_0 = a$ and let x_1 be the maximum of the set of all those $s \in [x_0, b]$ such that

$$\frac{\max_{x\in[x_0,s]} f(x) - \min_{x\in[x_0,s]} f(x)}{2} \le t.$$

As f is continuous, we have that

$$\frac{\max_{x\in[x_0,x_1]} f(x) - \min_{x\in[x_0,x_1]} f(x)}{2} = t.$$

Moreover, either $f(x_1) = \max_{x\in[x_0,x_1]} f(x)$ or $f(x_1) = \min_{x\in[x_0,x_1]} f(x)$. So, if we denote by c_1 the average value

$$c_1 = \frac{\max_{x\in[x_0,x_1]} f(x) + \min_{x\in[x_0,x_1]} f(x)}{2},$$

then we have that

$$\|f - c_1\|_{L^\infty[x_0,x_1]} \le t \quad \text{and} \quad |f(x_1) - c_1| = t.$$

Next we put

$$x_2 = \max\left\{s \in [x_1, b] : \frac{\max_{x\in[x_1,s]} f(x) - \min_{x\in[x_1,s]} f(x)}{2} \le t\right\},$$

and we let

$$c_2 = \frac{\max_{x\in[x_1,x_2]} f(x) + \min_{x\in[x_1,x_2]} f(x)}{2}.$$

We have now that

$$\|f - c_2\|_{L^\infty[x_1,x_2]} \le t \quad \text{and} \quad |f(x_2) - c_2| = t.$$

We continue this process choosing points $x_1, x_2, \ldots, x_k$ with $k = k(t) \ge 1$.

As the function f is continuous, it is uniformly continuous on $[a, b]$ and so there exists $\delta = \delta(t)$ such that $|f(x) - f(y)| \le \frac{1}{2}t$ if $|x - y| \le \delta$. It follows that $x_i - x_{i-1} \ge \delta$ for $i = 1, 2, \ldots, k$. Hence, the process ends after a finite number of steps with an $x_k \le b$.

If $x_k < b$ then we put $x_{k+1} = b$. In this case, we have

$$\frac{\max_{x\in[x_k,x_{k+1}]} f(x) - \min_{x\in[x_k,x_{k+1}]} f(x)}{2} < t.$$

We put

$$c_{k+1} = \begin{cases} \min_{x\in[x_k,x_{k+1}]} f(x) + t, & \text{if } f(x_k) = c_k - t \\ \max_{x\in[x_k,x_{k+1}]} f(x) - t, & \text{if } f(x_k) = c_k + t \end{cases}$$

and we have again

$$\|f - c_{k+1}\|_{L^\infty[x_k,x_{k+1}]} \le t.$$

To formulate the outcome, let $a = x_0 < x_1 < \cdots < x_N < x_{N+1} = b$ be the points that we have found, let (c_i) be the constants and let f_t be the piecewise constant function defined by

$$f_t = \sum_{i=1}^{N} c_i \chi_{[x_{i-1},x_i)} + c_{N+1}\chi_{[x_N,x_{N+1}]}. \tag{2.1}$$

Theorem 2.1. *Let f be a continuous function on the interval $[a, b]$ such that*

$$\frac{\max_{x\in[a,b]} f(x) - \min_{x\in[a,b]} f(x)}{2} > t.$$

Then for the function f_t defined by (2.1) we have

$$\|f - f_t\|_{L^\infty} \le t \quad \text{and} \quad \|f_t\|_{BV} = \inf_{\|g\|_{L^\infty}\le t} \|f - g\|_{BV}.$$

Consequently, the function $g_t = f - f_t$ is an exact minimizer for the E-functional of the couple (L^∞, BV).

Proof. The inequality $\|f - f_t\|_{L^\infty} \le t$ follows from the construction of f_t. So, we only need to prove that

$$\|f_t\|_{BV} = \inf_{\|g\|_{L^\infty}\le t} \|f - g\|_{BV}.$$

Since the value $\|f_t\|_{BV}$ is the sum of jumps, we have that

$$\|f_t\|_{BV} = \sum_{i=1}^{N} |c_{i+1} - c_i|.$$

Thus it suffices to check that for any function g with $\|g\|_{L^\infty} \le t$ we have that

$$\|f - g\|_{BV} \ge \sum_{i=1}^{N} |c_{i+1} - c_i|.$$

To prove this, we first note that $c_{i+1} \ne c_i$ because if they are equal then the construction would choose x_{i+1} instead of x_i, uniting the intervals $[x_{i-1}, x_i]$ and $[x_i, x_{i+1}]$.

If the sequence (c_i) is monotone decreasing,

$$c_1 > c_2 > \cdots > c_{N+1},$$

then it is easy to see that $f(x_1) = c_1 - t, \ldots, f(x_N) = c_N - t$. On the interval $[x_0, x_1]$ there exists a maximum point $\bar{x}_1$ of f such that

$$f(\bar{x}_1) = c_1 + t$$

and on the interval $[x_N, x_{N+1}]$ there is a minimum point $\tilde{x}_{N+1}$ of f with

$$f(\tilde{x}_{N+1}) = c_{N+1} - t.$$

Then $f(\bar{x}_1) - g(\bar{x}_1) \geq c_1$ and $f(\tilde{x}_{N+1}) - g(\tilde{x}_{N+1}) \leq c_{N+1}$. Hence,

$$\|f - g\|_{BV} \geq |f(\bar{x}_1) - g(\bar{x}_1) - f(\tilde{x}_{N+1}) + g(\tilde{x}_{N+1})| \geq c_1 - c_{N+1} = \sum_{i=1}^{N} |c_{i+1} - c_i| .$$

The case

$$c_1 < c_2 < \cdots < c_{N+1}$$

is similar. Now we have $f(x_1) = c_1 + t, \ldots, f(x_N) = c_N + t$. Choose in the interval $[x_0, x_1]$ a minimum point $\tilde{x}_1$ of f with

$$f(\tilde{x}_1) = c_1 - t$$

and pick in $[x_N, x_{N+1}]$ a maximum point $\bar{x}_{N+1}$ such that

$$f(\bar{x}_{N+1}) = c_{N+1} + t.$$

So, $f(\bar{x}_{N+1}) - g(\bar{x}_{N+1}) \geq c_{N+1}$ and $f(\tilde{x}_1) - g(\tilde{x}_1) \leq c_1$. Then it follows that

$$\|f - g\|_{BV} \geq |f(\tilde{x}_1) - g(\tilde{x}_1) - f(\bar{x}_{N+1}) + g(\bar{x}_{N+1})| \geq c_{N+1} - c_1 = \sum_{i=1}^{N} |c_{i+1} - c_i| .$$

Finally, in the general case, suppose that

$$c_1 > c_2 > \cdots > c_k \quad \text{and} \quad c_k < c_{k+1}.$$

Then, $c_1, c_2, \ldots, c_k$ is a maximal monotone part of the sequence $c_1, c_2, \ldots, c_{N+1}$. Proceeding as above, we get

$$f(x_1) = c_1 - t, \ldots, f(x_{k-1}) = c_{k-1} - t, \quad \text{and} \quad f(x_k) = c_k + t.$$

On the interval $[x_0, x_1]$ there exists a maximum point $\bar{x}_1$ of f such that

$$f(\bar{x}_1) = c_1 + t$$

and on $[x_{k-1}, x_k]$ there is a minimum point $\tilde{x}_k$ with

$$f(\tilde{x}_k) = c_k - t.$$

Then on the interval $[x_0, \tilde{x}_k]$ we have as above $f(\bar{x}_1) - g(\bar{x}_1) \geq c_1$ and $f(\tilde{x}_k) - g(\tilde{x}_k) \leq c_k$. Hence

$$\|f - g\|_{BV[x_0, \tilde{x}_k]} \geq |f(\bar{x}_1) - g(\bar{x}_1) - f(\tilde{x}_k) + g(\tilde{x}_k)| \geq c_1 - c_k = \|f_t\|_{BV[x_0, \tilde{x}_k]} .$$

To complete the proof we note that if we start the process from the point $\tilde{x}_k$ then we come to the same points $x_k, \ldots, x_{N+1}$ and the same constants $c_k, \ldots, c_{N+1}$ because $f(\tilde{x}_k) = c_k - t$ and $f(x_k) = c_k + t$. Since

$$\|f - g\|_{BV} = \|f - g\|_{BV[x_0, \tilde{x}_k]} + \|f - g\|_{BV[\tilde{x}_k, x_{N+1}]}$$

the required result follows by induction on the number of maximal monotonic parts in the sequence $c_1, c_2, \ldots, c_{N+1}$. $\square$

As a corollary of the construction we are going to obtain the announced formula for the K-functional for the couple (BV, L^∞). Recall that

$$K(N, f; BV, L^\infty) = \inf_{g \in L^\infty} \{\|f - g\|_{BV} + N\|g\|_{L^\infty}\}.$$

Theorem 2.2. *Let f be a continuous function on $[a, b]$. Then*

$$K(N, f; BV, L^\infty) \backsim \sup_{a \le x_0 \le x_1 \le \cdots \le x_N \le b} \sum_{i=1}^{N} |f(x_i) - f(x_{i-1})|.$$

Here, the constants in the equivalence are independent of $f \in C[a, b]$ and $N \in \mathbb{N}$.

Proof. Let

$$\varphi(N) = \sup_{a \le x_0 \le x_1 \le \cdots \le x_N \le b} \sum_{i=1}^{N} |f(x_i) - f(x_{i-1})|.$$

We claim that

$$K(N, f; BV, L^\infty) \ge \frac{1}{4}\varphi(N).$$

Indeed, if

$$\|g\|_{L^\infty} \ge \frac{1}{4N}\varphi(N)$$

then

$$\|f - g\|_{BV} + N\,\|g\|_{L^\infty} \ge N \cdot \frac{1}{4N}\varphi(N) = \frac{1}{4}\varphi(N).$$

Assume now that

$$\|g\|_{L^\infty} \le \frac{1}{4N}\varphi(N).$$

Take any $\varepsilon > 0$ and find points $a \le x_0 \le x_1 \le \cdots \le x_N \le b$ such that

$$\sum_{i=1}^{N} |f(x_i) - f(x_{i-1})| \ge \varphi(N) - \varepsilon.$$

Clearly

$$\sum_{i=1}^{N} |g(x_i) - g(x_{i-1})| \le 2N\,\|g\|_{L^\infty} \le \frac{1}{2}\varphi(N).$$

Therefore,

$$\|f - g\|_{BV} \ge \sum_{i=1}^{N} |(f(x_i) - g(x_i)) - (f(x_{i-1}) - g(x_{i-1}))|$$

$$\geq \sum_{i=1}^{N} |f(x_i) - f(x_{i-1})| - \sum_{i=1}^{N} |g(x_i) - g(x_{i-1})|$$
$$\geq \varphi(N) - \varepsilon - \frac{1}{2}\varphi(N) \geq \frac{1}{2}\varphi(N) - \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we derive that $\|f - g\|_{BV} \geq \frac{1}{2}\varphi(N)$. So in the case when $\|g\|_{L^\infty} \leq \frac{1}{4N}\varphi(N)$ we have that

$$\|f - g\|_{BV} + N\,\|g\|_{L^\infty} \geq \frac{1}{2}\varphi(N).$$

Consequently,

$$K(N, f; BV, L^\infty) = \inf_{g \in L^\infty} (\|f - g\|_{BV} + N\,\|g\|_{L^\infty}) \geq \frac{1}{4}\varphi(N).$$

To prove the converse inequality between the K-functional and φ, note that it follows from the definition of $\varphi(N)$ that $\varphi(2N) \leq 2\varphi(N)$. Write

$$t = \frac{\varphi(2N)}{N}$$

and apply to f the algorithm for the exact minimizer described above to find the points $x_0 = a < x_1 < \cdots < x_{N_t} < x_{N_t+1} = b$. On each of the intervals, except perhaps for the last one, the difference between the maximum and the minimum of f is $2t$. We claim that $N_t < N$. Indeed, if $N_t \geq N$ then we can find $2N$ points $y_1, y_2, \ldots, y_{2N}$ such that

$$\sum_{i=1}^{2N-1} |f(y_{i+1}) - f(y_i)| \geq 2t \cdot N = 2\varphi(2N)$$

which contradicts the definition of $\varphi(2N)$, so $N_t < N$. Using the function f_t given by the algorithm we get

$$K(N, f; BV, L^\infty) \leq \|f_t\|_{BV} + N\,\|f - f_t\|_{L^\infty}$$
$$\leq \|f_t\|_{BV} + N\frac{\varphi(2N)}{N} = \|f_t\|_{BV} + \varphi(2N).$$

Moreover

$$\|f_t\|_{BV} = \sum_{i=1}^{N_t} |c_{i+1} - c_i| \leq \sum_{i=1}^{N_t} (|c_i - f(x_i)| + |c_{i+1} - f(x_i)|)$$
$$\leq 2tN_t < 2tN \leq 2\varphi(2N).$$

Therefore,

$$K(N, f; BV, L^\infty) \leq 3\varphi(2N) \leq 6\varphi(N). \quad \square$$

3. The one-dimensional analogue of the Rudin–Osher–Fatemi model

Suppose that we observe a function ("noisy signal") $f(x)$ at M points $x_1 < x_2 < \cdots < x_M$ on the real line. Suppose also that

$$f = f_* + \eta$$

where f_* is the "initial data" ("initial signal") and η is the "noise" and that we know that

$$\max_{1\le i\le M} |\eta(x_i)| \le \varepsilon.$$

To put the problem in the framework that we have discussed above, we can take a as the point x_1, b as the point x_M, and we can extend the functions f, f_* and η to $[a, b]$ connecting the points $(x_i, f(x_i))$ by line segments, that is,

$$f(x) = f(x_i) + \frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i}(x - x_i), \quad x \in \left[x_i, x_{i+1}\right], \ 1 \le i \le M - 1,$$

with similar formulae for f_* and η. The problem that we want to face is to reconstruct approximately f_*.

If we apply the algorithm for constructing exact minimizer for the couple (L^∞, BV) with the level $t = \varepsilon$ then we obtain a function $f_\varepsilon \in BV[a, b]$ such that

$$\|f - f_\varepsilon\|_{L^\infty} \le \varepsilon \quad \text{and} \quad \|f_\varepsilon\|_{BV} \le \|f_*\|_{BV}.$$

Whence, proceeding as in the introduction, for any $\theta \in (0, 1)$ we obtain

$$\|f_* - f_\varepsilon\|_{L^\infty}^{1-\theta} \|f_* - f_\varepsilon\|_{BV}^{\theta} \le 2\varepsilon^{1-\theta} \|f_*\|_{BV}^{\theta}.$$

Now let us consider the space of p-variations defined by the seminorm

$$\|f\|_{BV_p} = \sup_{N\in\mathbb{N}} \sup_{a\le x_0\le\cdots\le x_N\le b} \left(\sum_{i=1}^{N} |f(x_i) - f(x_{i-1})|^p\right)^{\frac{1}{p}}, \quad 1 < p < \infty.$$

It follows from Hölder's inequality that

$$\left(\sum_{i=1}^{N} |f(x_i) - f(x_{i-1})|^p\right)^{\frac{1}{p}} \le \left(\max_i |f(x_i) - f(x_{i-1})|\right)^{\frac{p-1}{p}} \left(\sum_{i=1}^{N} |f(x_i) - f(x_{i-1})|\right)^{\frac{1}{p}}.$$

So, for any function $h \in BV$, we have

$$\|h\|_{BV_p} \le 2^{1-\frac{1}{p}} \|h\|_{L^\infty}^{1-\frac{1}{p}} \|h\|_{BV}^{\frac{1}{p}}.$$

Consequently, if we take $\theta = \frac{1}{p}$ then we obtain the following estimate of the error $f_\varepsilon - f_*$ in the space BV_p

$$\|f_\varepsilon - f_*\|_{BV_p} \le 2^{2-\frac{1}{p}} \varepsilon^{1-\frac{1}{p}} \|f_*\|_{BV}^{\frac{1}{p}}.$$

Note that we have realized Morozov's discrepancy principle in just one step. The reason for this is that we have an algorithm for constructing minimizers for the E-functional instead of the L-functional.

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